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# PROBABILISTIC ANALYSIS OF NON-SELF-ADJOINT MECHANICAL SYSTEMS WITH UNCERTAIN PARAMETERS

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**Abstract** – A new approach to analyze mechanical systems that have an uncertain distribution of material properties and loadings, for the variability of their dynamic response parameters is developed. The methodology is demonstrated through solving for the probabilistic moments of eigenvalues of a non-self-adjoint structural system. Material properties are modelled using random fields and uncertain loadings are modelled using random variables. By treating the random fluctuations of vibratory response to be the stochastic perturbations to mean response, second order moments of eigenvalues are obtained. To circumvent the practical difficulty of obtaining the exact correlation models for parameter variability, second order moments of eigenvalues are evaluated in terms of variance functions. Both the sensitivity of eigenvalues to the variability in system parameters, and the effects on the eigenvalue variability of correlation properties of uncertain parameters are demonstrated.

## 1. INTRODUCTION

The vibratory response of mechanical and structural systems is highly sensitive to parameter fluctuations. The dynamic response becomes a stochastic function when systems have an uncertain distribution of material properties and loadings. Most real-life loadings are random. The strength and deterioration characteristics of cost-effective materials such as composites, microgeometric topography of mechanical components etc., have an uncertain distribution due to manufacturing or measuring errors, variations in the sizes of fasteners and operating environment. In these circumstances, safer and reliable designs can be achieved only by probabilistic modelling and analysis (Hori, 1973; Ibrahim, 1987; Schueller and Shinozuka, 1987). Further, a probabilistic analysis is essential for the evaluation of reliability and service life of, and accumulated damage in a structure (Bogdanoff and Kozin, 1985; Bolotin, 1989).

Stochastic models for material properties, microgeometric topography of machined surfaces and environmental loadings have recently been developed (Shinozuka and Lenoe, 1976; Vanmarcke, 1983; Zhang and Kapoor, 1991). Vibration analysis of mechanical systems has been performed based on a probabilistic approach (Herrmann, 1971; Soong and Cozzarelli, 1976; Schueller and Shinozuka, 1987; Bucher and Brenner, 1992; Elishakoff and Colombi, 1993; Ramu and Ganesan, 1993). Eigenvalue problems of structural systems have been analyzed in the works of Boyce (1968), Shinozuka and Astill (1972), Soong and Cozzarelli (1976), Vom Scheidt and Purkert (1983), Ibrahim (1987), Benaroya (1991), and Ramu and Ganesan (1993). However, all these works deal only with self-adjoint systems i.e. structures subjected to conservative loadings.

Many important mechanical systems such as actively-controlled systems, rotor systems, aeroelastic structures and pipes conveying high velocity fluid flow, are described by non-self-adjoint differential equations. Rotor systems and non-conservatively loaded columns have been analyzed in the works of Ganesan *et al.* (1993) and Sankar *et al.* (1993). The uncertain axial loadings have not been considered in these works. Both the uncertain loadings as well as material properties have been considered in the work of Ramu and Ganesan (1992). However, (i) full covariance structure of the eigensolutions has not been obtained. (ii) only distributed axial loadings have been considered, (iii) only one boundary-condition has been considered, (iv) the expressions derived pose computational difficulties for certain types of correlation relationships and (v) the closed-form solutions developed are based on perturbation expansions that lead to less accurate results.

Further, in the above work, the stochastic fields have been characterized through their full-length correlation functions, which are seldom available in industrial practice. Newer measures of uncertainty such as scale of fluctuations (Vanmarcke, 1983) can be used to account for this practical difficulty. The expressions derived in the previous work (Ramu and Ganesan, 1992) have to be numerically integrated and they involve mixed functions of correlation functions and higher order derivatives of base solutions. Since the numerical method that is best suited for a particular correlation function need not be the best-suited one for the higher order derivatives of the base solutions, the computational accuracy is largely affected. Also, for certain correlation functions such as the singular functions of white noise field and the exponential functions of Markov processes and autoregressive correlation models, the numerical evaluation becomes highly inaccurate. All the above issues are considered in the present paper and a new approach that employs asymptotic expansions that are completely different from those employed in the previous works is developed. More general solutions that yield the complete covariance structure of eigen-solutions are developed in such a way that they can readily be employed in practical design and diagnostics problems. Both distributed and end axial loadings as well as different types of boundary conditions are considered in the present paper.

## 2. MATHEMATICAL FORMULATION

The non-self-adjoint structural system considered here is described by the following partial differential equation:

$$\{\bar{E}[1+a(x)]Iw''\}''+\bar{P}(1+c)w''+\bar{g}[1+d](L-x)w''+\bar{m}[1+b(x)]\ddot{w}=0. \quad (1)$$

Here $w$ is the characteristic state variable of the elastic system, primes denote partial differentiation with respect to $x$ and overdots denote partial differentiation with respect to $t$.

In the above equation, the second and third terms involving the second derivative of the state variable make the partial differential equation non-self-adjoint. Many important structural systems can be represented by this partial differential equation. The undamped-free motions of beam-columns subjected to axially-distributed as well as end follower forces and that of cylindrical pipes conveying fluids (the effect of coriolis accelerations is included separately) are described by the above equation. Further, the undamped-free motions of actively-controlled structures such as the antenna arm of a satellite, are described through the above equation. The first term in eqn (1) represents the effects of flexural rigidity of the structural system. The second and third terms represent, respectively, the effects of end thrust (or centrifugal forces in fluid-flowing pipes or control parameter in actively-controlled structures) and distributed axial loadings (or friction forces). The last term stands for the effects of inertial forces (structural, fluidic or coupled structure-fluid forces).

The boundary conditions are provided depending upon the type of the structural system being considered and further initial conditions about $w$ and $\delta w/\delta t$ at the time origin are also provided. Even though any set of boundary conditions can be taken into account by the present method of analysis, following two cases are considered in detail for the purposes of brevity and clarity:

*Case* I:

$$w(o,t) = w'(o,t) = 0 \quad (2)$$

$$w''(L,t) = \{\bar{E}[1+a(x)]I\times w''\}'|_{\mathrm{at}\,(L,t)} = 0 \quad (3)$$

*Case* II:

$$w(o, t) = w''(o, t) = 0 \tag{4}$$

$$w(L, t) = w''(L, t) = 0. \tag{5}$$

It may be noted here that when $P = 0$, eqns (1–3) describe the undamped-free motions of a rod subjected to distributed non conservative axial loadings, which is known as "Leipholz column". Similarly, when $g = 0$, eqns (1–3) model the undamped-free motions of (i) a rod subjected to follower force $P$ at the end, which is known as "Beck's column" and (ii) an antenna arm of a satellite that is actively controlled using a constant-feedback control factor $P$. In the above five equations, functions $a(x)$ and $b(x)$ are independent one-dimensional univariate homogeneous stochastic fields and further $c$ and $d$ are independent random variables. These stochastic coefficients are characterized as follows:

$a(x)$: Zero mean; variance $\sigma_a^2$; autocorrelation $R_{aa}(\tau)$; PSD function $S_{aa}(f)$; correlation function $P_{aa}(\tau)$; normalized PSD function $s_{aa}(f)$, scale of fluctuation $\Theta_a$.
$b(x)$: Zero mean; variance $\sigma_b^2$, autocorrelation $R_{bb}(\tau)$; PSD function $S_{bb}(f)$; correlation function $P_{bb}(\tau)$; normalized PSD function $s_{bb}(f)$; scale of fluctuation $\Theta_b$.
$c, d$: Zero means; respective variances $\sigma_c^2$ and $\sigma_d^2$.
$\tau, f$: The lag vector and the wave frequency.

The solution to eqn (1) is sought in the form

$$w(x, t) = X_n(x) \times T_n(t) \tag{6}$$

where $T_n(t)$ is a time function such that $\delta^2 T_n(t)/\delta t^2 = -\Omega_n^2 T_n(t)$ and it is implicit that the solution has a discrete spectrum. After substituting eqn (6) into eqn (1), one gets

$$1/L^4\{\bar{E}[1+a(\tau L)]IX_n''(\tau)\}''+1/L^2[X_n''(\tau)\bar{g}(1+d)(L-\tau L)]$$
$$+1/L^2\bar{P}(1+c)X_n''(\tau)-\bar{m}[1+b(\tau L)]\Omega_n^2 X_n(x) = 0 \tag{7}$$

where the dimensionless parameter $\tau = x/L$ is introduced so that primes now denote differentiation with respect to $\tau$. The boundary conditions correspondingly turn into

*Case* I:

$$X_n(o) = X_n'(o) = 0 \tag{8}$$

$$X_n''(1) = \{\bar{E}[1+a(\tau L)]I \times X_n''(\tau L)\}'|_{at\ \tau=1} = 0 \tag{9}$$

*Case* II:

$$X_n(o) = X_n''(o) = 0 \tag{10}$$

$$X(1) = X_n''(1) = 0. \tag{11}$$

After multiplying eqn (7) by $L^4$, dividing by $\bar{E}I$ and making the substitutions $G = L^2/\bar{E}I$, $\mu_n = \Omega_n^2 \times \bar{m}L^4/\bar{E}I$ and $g_l = \bar{g}L$, eqn (7) can be written as

$$\{[1+\alpha a(\tau)]X_n''(\tau)\}''+g_l G(1-\tau)(1+d)X_n''(\tau)+\bar{P}G(1+c)X_n''(\tau) = \mu_n[1+\beta b(\tau)]X_n(\tau). \tag{12}$$

In order to characterize the stochastic fields in the asymptotic analysis that follows, two perturbation parameters $\alpha$ and $\beta$ have been introduced into eqn (12) to be associated with $a(x)$ and $b(x)$ respectively. The following asymptotic expansions are now employed so as lead to the determination of full covariance structure of the response variables in an easier and more straightforward manner.

$$\mu_n = \mu_{n0} + \alpha\mu_{n1} + \beta\mu_{n2} + d\mu_{n3} + c\mu_{n4} + \ldots \tag{13}$$

$$X_n(\tau) = X_{n0}(\tau) + \alpha X_{n1}(\tau) + \beta X_{n2}(\tau) + dX_{n3}(\tau) + cX_{n4}(\tau) + \ldots. \tag{14}$$

It may be noted that the above asymptotic series expansions are different from that of the previous work by the present author (Ramu and Ganesan, 1992) in that the stochastic coefficients of the partial differential equation, $c$ and $d$ (which are the random variables that correspond to constant end thrust and distributed axial loading respectively) are being used as the perturbation parameters of the asymptotic series expansions that correspond to eigensolutions. It is this form of asymptotic expansions that actually leads to the calculation of the covariance structure of eigenvalues in an easier and more accurate manner than that of the work by Ramu and Ganesan (1992).

After substituting the above two asymptotic expansions in eqn (12), one gets

$$\begin{aligned}
&\{[1+\alpha a(\tau)][X''_{n0}(\tau)+\alpha X''_{n1}(\tau)+\beta X''_{n2}(\tau)+dX''_{n3}(\tau)+cX''_{n4}(\tau)+\ldots]\}'' \\
&\quad + g_l G(1-\tau)(1+d)\{X''_{n0}(\tau)+\alpha X''_{n1}(\tau)+\beta X''_{n2}(\tau)+dX''_{n3}(\tau)+cX''_{n4}(\tau)+\ldots\} \\
&\quad + \bar{P}G(1+c)\{X''_{n0}(\tau)+\alpha X''_{n1}(\tau)+\beta X''_{n2}(\tau)+dX''_{n3}(\tau)+cX''_{n4}(\tau)+\ldots\} \\
&= \{\mu_{n0}+\alpha\mu_{n1}+\beta\mu_{n2}+d\mu_{n3}+c\mu_{n4}+\ldots\} \\
&\quad \{X_{n0}(\tau)+\alpha X_{n1}(\tau)+\beta X_{n2}(\tau)+dX_{n3}(\tau)+cX_{n4}(\tau)+\ldots\}\{1+\beta b(\tau)\}.
\end{aligned} \tag{15}$$

On expanding eqn (15), collecting terms of like power in $\alpha$, $\beta$, $d$ and $c$ and then setting $\alpha = \beta = 1$, a set of ordinary differential equations are obtained for $X_{ni}(\tau)$, $i = 0, 1, 2, \ldots$:

$$X''''_{n0}(\tau) + g_l G X''_{n0}(\tau) - g_l G\tau X''_{n0}(\tau) + PGX''_{n0}(\tau) = \mu_{n0}X_{n0}(\tau) \tag{16}$$

$$\begin{aligned}
X''''_{n1}(\tau) + (g_l G - g_l G\tau)X''_{n1}(\tau) + a(\tau)X''''_{n0}(\tau) + PGX''_{n1}(\tau) + 2a'(\tau)X'''_{n0}(\tau) + a''(\tau)X''_{n0}(\tau) \\
= \mu_{n0}X_{n1}(\tau) + \mu_{n1}X_{n0}(\tau)
\end{aligned} \tag{17}$$

$$X''''_{n2}(\tau) + (g_l G - g_l G\tau)X''_{n2}(\tau) + PGX''_{n2}(\tau) = \mu_{n0}X_{n2}(\tau) + \mu_{n2}X_{n0}(\tau) + \mu_{n0}b(\tau)X_{n0}(\tau) \tag{18}$$

$$X''''_{n3}(\tau) + (g_l G - g_l G\tau)(X''_{n3}(\tau) + X''_{n0}(\tau)) + PGX''_{n3}(\tau) = \mu_{n0}X_{n3}(\tau) + \mu_{n3}X_{n0}(\tau) \tag{19}$$

$$X''''_{n4}(\tau) + (g_l G - g_l G\tau)X''_{n4}(\tau) + PG(X''_{n4}(\tau) + X''_{n0}(\tau)) = \mu_{n4}X_{n0}(\tau) + \mu_{n0}X_{n4}(\tau)\ldots. \tag{20}$$

In the above equations, overbars associated with $P$ have been dropped for convenience. Corresponding to the above set of differential equations and asymptotic expansions given by eqns (13) and (14), the boundary conditions are rewritten as follows:

| *Case* I | *Case* II | |
|---|---|---|
| $X_{n0}(0) = 0$ | $X_{n0}(0) = 0$ | (21) |
| $X'_{n0}(0) = 0$ | $X''_{n0}(0) = 0$ | (22) |
| $X''_{n0}(1) = 0$ | $X_{n0}(1) = 0$ | (23) |
| $X'''_{n0}(1) = 0$ | $X''_{n0}(1) = 0$ | (24) |

$$Case\,\text{I}: X_{n1}(0) = X_{n2}(0) = X_{n3}(0) = X_{n4}(0) = X'_{n1}(0) = X'_{n2}(0) = X'_{n3}(0) = X'_{n4}(0) = 0, \tag{25}$$

$$Case\,\text{II}: X_{n1}(0) = X_{n2}(0) = X_{n3}(0) = X_{n4}(0) = X''_{n1}(0) = X''_{n2}(0) = X''_{n3}(0) = X''_{n4}(0) = 0 \tag{26}$$

$$Case\,\text{I}: X''_{n1}(1) = X''_{n2}(1) = X''_{n3}(1) = X''_{n4}(1) = X'''_{n1}(1) = X'''_{n2}(1) = X'''_{n3}(1) = X'''_{n4}(1) = 0$$

$$Case\,\text{II}: X_{n1}(1) = X_{n2}(1) = X_{n3}(1) = X_{n4}(1) = X''_{n1}(1) = X''_{n2}(1) = X''_{n3}(1) = X''_{n4}(1) = 0 \tag{27}$$

The generating solution $\mu_{n0}$ and $X_{n0}(\tau)$ are obtained from eqn (16) corresponding to boundary conditions given by eqns (21)–(24). Using these, other components of the asymptotic series are obtained as follows. Considering eqn (18), after multiplying by $X_{n0}(\tau)$ and then integrating between 0 and 1, $\mu_{n1}$ can be obtained corresponding to any set of sample realizations of stochastic parameters $a(\tau)$, $b(\tau)$, $c$ and $d$, according to:

$$\mu_{n1} = \frac{\int_0^1 a''(\tau)X''_{n0}(\tau)X_{n0}(\tau)\,\mathrm{d}\tau + 2\int_0^1 a'(\tau)X'''_{n0}(\tau)X_{n0}(\tau)\,\mathrm{d}\tau + \int_0^1 a(\tau)X''''_{n0}(\tau)X_{n0}(\tau)\,\mathrm{d}\tau}{\int_0^1 X^2_{n0}(\tau)\,\mathrm{d}\tau}. \tag{28}$$

It may be noted that point fluctuations of the coefficients of the non-self-adjoint system are considered to constitute the stochastic perturbations in the asymptotic series expansions for eigensolutions. After integrating by parts and using the boundary conditions for $X_{n0}(\tau)$, $\mu_{n1}$ is approximated as,

$$\mu_{n1} = \int_0^1 a(\tau)[X''_{n0}(\tau)]^2\,\mathrm{d}\tau \left\{\int_0^1 X^2_{n0}(\tau)\,\mathrm{d}\tau\right\}. \tag{29}$$

In a similar manner other components of the asymptotic expansion for eigenvalues are obtained and they are given by

$$\mu_{n2} = -\mu_{n0}\int_0^1 b(\tau)X^2_{n0}(\tau)\,\mathrm{d}\tau \left\{\int_0^1 X^2_{n0}(\tau)\,\mathrm{d}\tau\right\} \tag{30}$$

$$\mu_{n3} = g_L G\int_0^1 (1-\tau)X''_{n0}(\tau)X_{n0}(\tau)\,\mathrm{d}\tau \left\{\int_0^1 X^2_{n0}(\tau)\,\mathrm{d}\tau\right\} \tag{31}$$

$$\mu_{n4} = PG\int_0^1 X''_{n0}(\tau)X_{n0}(\tau)\,\mathrm{d}\tau \left\{\int_0^1 X^2_{n0}(\tau)\,\mathrm{d}\tau\right\}. \tag{32}$$

From eqns (13) and (28)–(32) it can be shown that the mean value of $\mu_n$ is equal to $\mu_{n0}$ since the stochastic fields $a(x)$ and $b(x)$ are zero-mean fields and further $c$ and $d$ are zero-mean random variables. The expressions for covariance between any two eigenvalues $\mu_i$ and $\mu_j$, and variance of any eigenvalue $\mu_i$ are now evaluated.

$$\mathrm{Cov}(\mu_i, \mu_j) = \langle (\mu_i - \bar{\mu}_i)\quad(\mu_j - \bar{\mu}_j) \rangle = \left\{ \int_0^1 X_{i0}^2(\tau)\,\mathrm{d}\tau \int_0^1 X_{j0}^2(\tau)\,\mathrm{d}\tau \right\}^{1}$$

$$\times \left\{ \int_0^1 \int_0^1 R_{aa}(\tau_1 - \tau_2)[X''_{i0}(\tau_1)]^2 [X''_{j0}(\tau_2)]^2\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2 \right.$$

$$+ \mu_{i0}\mu_{j0} \int_0^1 \int_0^1 R_{bb}(\tau_1 - \tau_2) X_{i0}^2(\tau_1) X_{j0}^2(\tau_2)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2$$

$$+ (g_L G)^2 \sigma_d^2 \int_0^1 (1 - \tau_1) X''_{i0}(\tau_1) X_{i0}(\tau_1)\,\mathrm{d}\tau_1 \int_0^1 (1 - \tau_2) X''_{j0}(\tau_2) X_{j0}(\tau_2)\,\mathrm{d}\tau_2$$

$$\left. + (PG)^2 \sigma_c^2 \int_0^1 X''_{i0}(\tau_1) X_{i0}(\tau_1)\,\mathrm{d}\tau_1 \int_0^1 X''_{j0}(\tau_2) X_{j0}(\tau_2)\,\mathrm{d}\tau_2 + \ldots \right\}. \tag{33}$$

$$\mathrm{Var}(\mu_i) = \left\{ 1 \quad \left[ \int_0^1 X_{i0}^2(\tau)\,\mathrm{d}\tau \right]^2 \right\} \left\{ \int_0^1 \int_0^1 R_{aa}(\tau_1 - \tau_2)[X''_{i0}(\tau_1)]^2 [X''_{i0}(\tau_2)]^2\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2 \right.$$

$$+ \mu_{i0}^2 \int_0^1 \int_0^1 R_{bb}(\tau_1 - \tau_2) X_{i0}^2(\tau_1) X_{i0}^2(\tau_2)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2 + (g_L G)^2 \sigma_d^2 \left[ \int_0^1 (1 - \tau) X''_{i0}(\tau) X_{i0}(\tau)\,\mathrm{d}\tau \right]^2$$

$$\left. + (PG)^2 \sigma_c^2 \left[ \int_0^1 X''_{i0}(\tau) X_{i0}(\tau)\,\mathrm{d}\tau \right]^2 + \ldots \right\}. \tag{34}$$

It can be seen from the above two equations that the eigenvalue variability due to the mass density is often more than that due to the Young's modulus, i.e. the eigenvalues are more sensitive to variations in mass density than that in Young's modulus. It can also be observed that for particular values of covariances of random fields $a(x)$ and $b(x)$ and random variables $c$ and $d$, it is the shape of the correlation functions or corresponding power spectral density functions that determines the values of covariances of eigenvalues. The correlation function could be an exponential decay function which corresponds to first order autoregressive model or any other linear, quadratic or sine functions. The evaluation of a suitable correlation model from among various competing models using the experimental data is a tedious job and in most cases, a satisfactory model would not have been evaluated at all. An alternative is given here for use in this circumstance that makes use of the theory of spatial averages. The stochastic coefficients of the differential system are now characterized through the scale of fluctuations or variance functions which are the more-recently-developed measures of stochasticity. A detailed theory about the theory of spatial averages can be seen in Vanmarcke (1983). The spatial averages of stochastic coefficients are formed over the domain of the differential equation and are obtained as

$$a_L = (1/L) \int_0^1 a(\tau)\,\mathrm{d}\tau; \quad b_L = (1/L) \int_0^1 b(\tau)\,\mathrm{d}\tau. \tag{35,36}$$

Probabilistic moments of these spatial averages are given below:

$$\bar{a}_L = \bar{b}_L = 0 \tag{37}$$

$$\mathrm{Var}(a_L) = \sigma_a^2 / L^2 \int_0^L \int_0^L P_{aa}(\tau_1 - \tau_2)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2 \tag{38}$$

$$\mathrm{Var}\,(b_l) = \sigma_b^2\, L^2 \int_0^l \int_0^l \rho_{bb}(\tau_1 - \tau_2)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2 \tag{39}$$

If $\Gamma_a\,(L)$ and $\Gamma_b\,(L)$ denote respectively the variance functions determined from the respective scales of fluctuations of the random fields $a(x)$ and $b(x)$, the above two equations can be rewritten as,

$$\mathrm{Var}\,(a_l) = \sigma_a^2\Gamma_a(L); \quad \mathrm{Var}\,(b_l) = \sigma_b^2\Gamma_b(L). \tag{40,41}$$

Now, based on local averages of stochastic coefficients, second order moments of eigenvalues are given below:

$$\begin{aligned}\mathrm{Cov}\,(\mu_i, \mu_j) = &\left\{\int_0^1 X_{i0}^2(\tau)\,\mathrm{d}\tau \int_0^1 X_{j0}^2(\tau)\,\mathrm{d}\tau\right\}^{-1} \left\{\sigma_a^2\Gamma_a(L) \int_0^1 [X''_{i0}(\tau_1)]^2\,\mathrm{d}\tau_1 \int_0^1 [X''_{j0}(\tau_2)]^2\,\mathrm{d}\tau_2\right.\\ &+ \mu_{i0}\mu_{j0}\sigma_b^2 \times \Gamma_b(L) \int_0^1 X_{i0}^2(\tau_1)\,\mathrm{d}\tau_1 \int_0^1 X_{j0}^2(\tau_2)\,\mathrm{d}\tau_2\\ &+ (g_l G)^2\sigma_a^2 \int_0^1 (1-\tau_1)X''_{i0}(\tau_1)X_{i0}(\tau_1)\,\mathrm{d}\tau_1 \int_0^1 (1-\tau_2)X''_{j0}(\tau_2)X_{j0}(\tau_2)\,\mathrm{d}\tau_2\\ &\left.+ (PG)^2\sigma^2 \int_0^1 X''_{i0}(\tau_1)X_{i0}(\tau_1)\,\mathrm{d}\tau_1 \int_0^1 X''_{j0}(\tau_2)X_{j0}(\tau_2)\,\mathrm{d}\tau_2 + \ldots\right\}.\end{aligned} \tag{42}$$

Now, it can be observed that the evaluation of second order moments of eigenvalues can be carried out in a more accurate manner in that each integration involves only one particular type of function. The numerical method that is best suited for a particular integrand can be employed in each case thus leading to the overall accuracy. Further, integrals that involve products of correlation functions and eigenfunctions are not present in the covariance expression for eigenvalues derived herein.

## 3. APPLICATIONS

(i) *Fluid conveying pipes*

The homogeneous equation of motion of a cantilevered tubular pipe of length $L$ conveying fluid with a deterministic velocity $U$ and having a stochastic distribution of Young's modulus and mass density is given as,

$$\{\bar{E}[1 + a(x)]Iw''\}'' + Fw'' + \bar{m}[1 + b(x)]\ddot{w} + 2MU\dot{w}' + M\ddot{w} = 0 \tag{43}$$

subject to the boundary conditions,

$$w(0, t) = w'(0, t) = w''(L, t) = \{\bar{E}[1 + a(x)]Iw''\}'|_{x=L} = 0. \tag{44}$$

In the above equations, $E$ and $m$ denote respectively the Young's modulus and mass per unit length of the pipe material, $M$ is the mass per unit length of the flowing fluid and $F$ is the centrifugal force. This equation describes the free, undamped oscillations of the system neglecting the effect of gravity forces. The centrifugal force is denoted by $F$ and further $F = MU^2$. The corresponding adjoint system can be written by the equation,

$$\{\bar{E}[1 + a(x)]Iw''\}'' + Fw'' + \bar{m}[1 + b(x)]\ddot{w} - 2MU\dot{w}' + M\ddot{w} = 0 \tag{45}$$

subject to the boundary conditions,

$$w(0,t) = w'(0,t) = 0;\quad \bar{E}[1 + a(x)]Iw''(x,t) + Fw(x,t) = 0 \quad \text{at } x = L;$$

$$\{\bar{E}[1 - a(x)]Iw''(x,t)\}' + Fw'(x,t) = 0 \text{ at } x = L. \tag{46}$$

The second order moments of eigenvalues can be obtained using the methods described in the previous sections, after some modifications. From eqn (33), the following expression is obtained:

$$\mathrm{Cov}(\mu_i, \mu_j) = \frac{1}{(1+\eta)^2 \left\{\int_0^1 X_{i0}^2(\tau)\,d\tau \int_0^1 X_{j0}^2(\tau)\,d\tau\right\}} \left\{\sigma_a^2 \int_0^1 \int_0^1 P_{aa}(\tau_1 - \tau_2)[X''_{i0}(\tau_1)]^2 \right.$$

$$\left. \times [X''_{j0}(\tau_2)]^2\,d\tau_1\,d\tau_2 + \mu_i \mu_j \sigma_b^2 \int_0^1 \int_0^1 P_{bb}(\tau_1 - \tau_2) X_{i0}^2(\tau_1) X_{j0}^2(\tau_2)\,d\tau_1\,d\tau_2 \right\} \tag{47}$$

where the dimensionless parameter $\eta$ is equal to the ratio $M/m$. In terms of variance functions, the covariance between the eigenvalues is written as

$$\mathrm{Cov}(\mu_i, \mu_j) = \frac{1}{(1+\eta)^2 \left\{\int_0^1 X_{i0}^2(\tau)\,d\tau \int_0^1 X_{j0}^2(\tau)\,d\tau\right\}} \left\{\sigma_a^2 \times \Gamma_a(L) \int_0^1 (X''_{i0}(\tau))^2\,d\tau \right.$$

$$\left. \times \int_0^1 [X''_{j0}(\tau)]^2\,d\tau + \mu_i \mu_j \sigma_b^2 \Gamma_b(L) \int_0^1 X_{i0}^2(\tau)\,d\tau \int_0^1 X_{j0}^2(\tau)\,d\tau \right\}. \tag{48}$$

(ii) *Non conservatively loaded columns*

First, a cantilever column loaded by a tip follower force at one end, which is known as Beck's column, is considered. Young's modulus $E$, mass density $m$ and follower load $P$ are random. The undamped, free oscillations are given by the differential equation,

$$\{\bar{E}[1 + a(x)]Iw''\}'' + \bar{P}(1 + c)w'' + \bar{m}[1 + b(x)]\ddot{w} = 0 \tag{49}$$

subject to the boundary conditions,

$$w(0,t) = w'(0,t) = 0;\quad w''(L,t) = 0;\ \{\bar{E}I[1 + a(x)]w''\}' = 0 \quad \text{at } x = L. \tag{50}$$

The adjoint system is given by,

$$\{\bar{E}[1 + a(x)]Iw''\}'' + \bar{P}(1 + c)w'' + \bar{m}[1 + b(x)]\ddot{w} = 0 \tag{51}$$

subject to the boundary conditions,

$$w(0,t) = w'(0,t) = 0;\quad \bar{E}I(1 + a(x))w'' + \bar{P}(1 + c)w = 0 \quad \text{at } x = L;$$

$$\{\bar{E}I[1 + a(x)]w''\}' + \bar{P}(1 + c)w' = 0 \quad \text{at } x = L. \tag{52}$$

Using these differential equations, the second moments of vibration frequencies are given as

$$\operatorname{Var}(\mu_i) = \frac{1}{\left\{\int_0^1 X_{i0}^2(\tau)\,\mathrm{d}\tau\right\}^2}\left\{\sigma_a^2 \int_0^1\int_0^1 P_{aa}(\tau_1-\tau_2)[X''_{i0}(\tau_1)]^2\right.$$

$$\times [X''_{i0}(\tau_2)]^2\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2 + \mu_{i0}^2\sigma_b^2 \int_0^1\int_0^1 P_{bb}(\tau_1-\tau_2)X_{i0}^2(\tau_1)X_{i0}^2(\tau_2)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2$$

$$\left. + (\bar{P}G)^2\sigma_c^2\left[\int_0^1 X''_{i0}(\tau_1)X_{i0}(\tau_1)\,\mathrm{d}\tau_1\right]^2\right\} \quad (53)$$

$$\operatorname{Cov}(\mu_i,\mu_j) = \frac{1}{\left\{\int_0^1 X_{i0}^2(\tau)\,\mathrm{d}\tau \int_0^1 X_{j0}^2(\tau)\,\mathrm{d}\tau\right\}}$$

$$\times\left\{\sigma_a^2\Gamma_a(L)\int_0^1 (X''_{i0}(\tau_1))^2\,\mathrm{d}\tau_1 \int_0^1 (X''_{j0}(\tau_2))^2\,\mathrm{d}\tau_2 + \mu_{i0}\cdot\mu_{j0}\sigma_b^2\Gamma_b(L)\int_0^1 X_{i0}^2(\tau_1)\,\mathrm{d}\tau_1\int_0^1 X_{j0}^2(\tau_2)\,\mathrm{d}\tau_2\right.$$

$$\left. + (\bar{P}G)^2\sigma_c^2 \int_0^1 X''_{i0}(\tau_1)X_{i0}(\tau_1)\,\mathrm{d}\tau_1 \cdot \int_0^1 X''_{j0}(\tau_2)X_{j0}(\tau_2)\,\mathrm{d}\tau_2 \cdot\right\}. \quad (54)$$

## 4. NUMERICAL EXAMPLE

First a Leipholz column of one metre length is considered. Introduced is the parameter $g^* = g/EI$, for generalization. Following five correlation models are considered to characterize the second order statistical properties of the system.

1. The triangular correlation function given by

$$P(\tau_1-\tau_2) = \begin{cases} 1 - \dfrac{|\tau_1-\tau_2|}{\tilde{a}}, & |\tau_1-\tau_2| < \tilde{a} \\ 0, & |\tau_1-\tau_2| \geqslant \tilde{a} \end{cases}$$

2. The first-order autoregressive random field correlation, (commonly known as AR (1) of Box-Jenkins models) given by

$$P(\tau_1-\tau_2) = \mathrm{e}^{-|\tau_1-\tau_2|/\tilde{b}}, \quad \tilde{b} = \text{a constant} = f(\varepsilon)$$

3. Second-order autoregressive correlation model given as,

$$P(\tau_1-\tau_2) = \left[1 + \frac{|\tau_1-\tau_2|}{\tilde{c}}\right]\mathrm{e}^{-|\tau_1-\tau_2|/\tilde{c}}$$

4. Squared exponential correlation model commonly known as Gaussian correlation model, given by

$$P(\tau_1-\tau_2) = \mathrm{e}^{-((\tau_1-\tau_2)/\tilde{d})^2}$$

5. Sine function correlation model that is given by

$$P(\tau_1 - \tau_2) = 2s_0 \frac{\sin f_u(\tau_1 - \tau_2)}{(\tau_1 - \tau_2)}$$

where, $s_0$ is the strength of white noise and $f_u$ is the upper cut-off frequency of the power spectral density given by

$$S(f) = S_0 = \begin{cases} \dfrac{\sigma^2}{2f_u}, & |f| < f_u \\ 0, & |f| > f_u \end{cases}.$$

It may be noted that this correlation model pertains (Shinozuka, 1987) to a mean-zero band-limiting white noise process in which the bandwidth increases without limit. This correlation model has been shown to be highly useful in obtaining upper and lower bounds for both parameter and response variability of stochastic structural systems.

In the above expressions, $\tilde{a}$, $\tilde{b}$, $\tilde{c}$, and $\tilde{d}$ denote parameters of the correlation models. The mass density and Young's modulus are considered to be the uncertain parameters. The variances of fundamental free vibration frequency for different correlation structures of the input fields and input variances are given in Table 1. In all these cases, $\tilde{a} = \tilde{b} = \tilde{c} = \tilde{d} = 25$ and $S_0 = 0.01$. This study gives the effect of randomness in different system parameters on the free vibration frequency statistics. To study the effect of the constants of the correlation models, the results for different sets of values of $a$ and $b$ are given in Table 2. For a particular set of constants of correlation models, the effect of type of correlation models on the

Table 1. Variances of $\mu_1$ for $g^* = 30$ when $E$ and $m$ are random fields $\mu_1 = 25$; $a = b = c = d = 25$

| Input variance $\sigma_a^2 = \sigma_b^2$ | Triangular | Correlation function First-order AR | Second-order AR | Gaussian | Finite power white noise $f_u = 20$ $S_0 = 10^{-2}$ |
|---|---|---|---|---|---|
| $1 \times 10^{-2}$ | 7.5092 | 7.5100 | 6.9397 | 7.5092 | 7.509 |
| $2 \times 10^{-2}$ | 15.0179 | 15.0196 | 13.8769 | 15.0719 | 15.071 |
| $3 \times 10^{-2}$ | 22.5268 | 22.5293 | 20.7847 | 22.5268 | 22.526 |
| $4 \times 10^{-2}$ | 30.0358 | 30.0391 | 27.6650 | 30.0358 | 30.035 |
| $5 \times 10^{-2}$ | 37.5447 | 37.5489 | 34.5812 | 37.5447 | 37.544 |
| $6 \times 10^{-2}$ | 45.0537 | 45.0587 | 41.4975 | 45.0537 | 45.053 |
| $7 \times 10^{-2}$ | 52.5626 | 52.5685 | 48.4030 | 52.5626 | 52.562 |
| $8 \times 10^{-2}$ | 60.0715 | 60.0783 | 55.2889 | 60.0715 | 60.071 |
| $9 \times 10^{-2}$ | 67.5805 | 67.5880 | 62.2000 | 67.5805 | 67.580 |
| $10 \times 10^{-2}$ | 75.0894 | 75.0978 | 69.0514 | 75.0894 | 75.089 |

Table 2. Effect of constants of correlation models on var $(\mu_1)$ when $E$ and $m$ are random fields

| Input variance | Triangular correlation | | First-order AR | |
|---|---|---|---|---|
| $\sigma_a^2 = \sigma_b^2 = \sigma^2$ | For $a = 15$ | For $a = 25$ | For $b = 25$ | For $b = 15$ |
| $1 \times 10^{-2}$ | 7.4508 | 7.5092 | 7.5100 | 7.4531 |
| $2 \times 10^{-2}$ | 14.9010 | 15.0179 | 15.0196 | 14.9057 |
| $3 \times 10^{-2}$ | 22.3515 | 22.5268 | 22.5293 | 22.3585 |
| $4 \times 10^{-2}$ | 29.8020 | 30.0358 | 30.0391 | 29.8113 |
| $5 \times 10^{-2}$ | 37.2525 | 37.5447 | 37.5489 | 37.2541 |
| $6 \times 10^{-2}$ | 44.7030 | 45.0537 | 45.0587 | 44.7170 |
| $7 \times 10^{-2}$ | 52.1535 | 52.5626 | 52.5685 | 52.1698 |
| $8 \times 10^{-2}$ | 59.6040 | 60.0715 | 60.0783 | 59.6226 |
| $9 \times 10^{-2}$ | 67.0545 | 67.5805 | 67.5880 | 67.0754 |
| $10 \times 10^{-2}$ | 74.5050 | 75.0894 | 75.0978 | 74.5283 |

variances of $\mu_1$ is now sought. Two cases are considered: (i) when only $E$ is random (case B) and (ii) when both $E$ and $m$ are random (case A). The results are given in Table 3.

Now, a cantilever pipe of 1 metre length which has a stochastically distributed $E$ and $m$ is considered. Flow velocity of the fluid that flows through the pipe is taken to be deterministic. For the mean problem, the method suggested in Paidoussis (1970) is adopted. First, $E$ alone is treated to be random. The variances of $\mu_1$, for the five correlation models given in the first example are given in Table 4. Now both $E$ and $m$ are treated to be random and the results are given in Table 5. Symbols are used in accordance with Paidoussis (1970). Further results are plotted in Figs 1 and 2.

These numerical results show the impact of randomness of each of the system parameters on the response variability and also the dependence of response moments on the type of the correlation models being employed to account for the parameter variability. Also, the dependence of response moments on the length of correlation of the random fields

Table 3. Var ($\mu_1$) for different input variances $a = b = c = 25$

| Input variance for random fields | Triangular Case A | Triangular Case B | First-order AR Case A | First-order AR Case B | Second-order AR Case A | Second-order AR Case B |
|---|---|---|---|---|---|---|
| 0.10 | 75.2739 | 1.0900 | 75.2823 | 1.0901 | 69.2235 | 1.0350 |
| 0.11 | 82.8012 | 1.1990 | 82.8105 | 1.1991 | 76.0992 | 1.1385 |
| 0.12 | 90.2286 | 1.3080 | 90.3387 | 1.3081 | 83.0173 | 1.2420 |
| 0.13 | 97.8560 | 1.4170 | 97.8669 | 1.4171 | 89.9354 | 1.3485 |
| 0.14 | 105.3834 | 1.2560 | 105.3952 | 1.5261 | 96.8535 | 1.4490 |
| 0.15 | 112.9108 | 1.6350 | 112.9234 | 1.6351 | 103.7716 | 1.5525 |
| 0.16 | 120.4382 | 1.7440 | 120.4516 | 1.7441 | 110.6897 | 1.6560 |
| 0.17 | 127.9656 | 1.8530 | 127.9797 | 1.8531 | 117.6078 | 1.7595 |
| 0.18 | 135.4929 | 1.9620 | 135.5081 | 1.9621 | 124.5259 | 1.8630 |
| 0.19 | 145.0203 | 2.0710 | 143.0363 | 2.0712 | 131.4440 | 1.9665 |

Table 4. Variances of $\mu_1$ for $U = 5.5$ and $\beta = 0.2$, when $E$ is a random field $a = b = c = d = 15$

| Input variance $\sigma_a^2 = \sigma^2$ | Correlation model: Triangular | First-order AR | Second-order AR | Gaussian | Finite power white noise $f_u = 10$, $s_0 = 10^{-3}$ |
|---|---|---|---|---|---|
| 0.10 | 4012.0289 | 4012.6018 | 3668.4460 | 4012.0289 | 4012.028 |
| 0.11 | 4413.2318 | 4413.8620 | 4034.9824 | 4413.2318 | 4413.231 |
| 0.12 | 4814.4347 | 4815.1222 | 4401.7989 | 4814.4347 | 4814.434 |
| 0.13 | 5215.6376 | 5216.3823 | 4768.4328 | 5215.6376 | 5215.637 |
| 0.14 | 5616.8404 | 5617.6425 | 5135.1812 | 5616.8404 | 5616.840 |
| 0.15 | 6018.0434 | 6018.9028 | 5501.7430 | 6018.0434 | 6018.043 |
| 0.16 | 6419.2462 | 6420.1629 | 5868.5258 | 6419.2462 | 6419.246 |
| 0.17 | 6820.4492 | 6821.4231 | 6235.3088 | 6820.4492 | 6820.449 |
| 0.18 | 7221.6520 | 7222.6832 | 6602.0916 | 7221.6520 | 7221.652 |
| 0.19 | 7622.7130 | 7623.815 | 6968.8743 | 7622.7130 | 7622.713 |

Table 5. Variances of $\mu_1$ for $U = 5.5$ and $\beta = 0.2$, when $E$ and $m$ are random fields. $a = b = c = d = 15$

| Input variance $\sigma_a^2 = \sigma_b^2 = \sigma^2$ | Correlation model: Triangular | First-order AR | Second-order AR | Gaussian | Finite power white noise $f_u = 10$, $S_0 = 10^{-3}$ |
|---|---|---|---|---|---|
| 0.10 | 4043.7961 | 4044.3706 | 3697.9766 | 4043.7961 | 4043.7961 |
| 0.11 | 4448.1757 | 4448.8076 | 4067.4691 | 4448.1757 | 4448.1757 |
| 0.12 | 4852.5553 | 4853.2447 | 4437.2312 | 4852.5553 | 4852.5553 |
| 0.13 | 5256.9349 | 5257.6817 | 4806.8146 | 5256.9349 | 5256.9349 |
| 0.14 | 5661.3145 | 5662.1188 | 5176.5183 | 5661.3145 | 5661.3145 |
| 0.15 | 6065.6942 | 6066.5559 | 5546.0255 | 6065.6942 | 6065.6942 |
| 0.16 | 6470.0737 | 6470.9929 | 5915.7605 | 6470.0737 | 6470.0737 |
| 0.17 | 6874.4554 | 6875.4300 | 6285.4956 | 6874.4534 | 6874.4534 |
| 0.18 | 7278.8329 | 7279.8670 | 6655.2305 | 7278.8329 | 7278.8329 |
| 0.19 | 7683.0689 | 7684.1604 | 7024.9655 | 7683.0689 | 7683.0689 |

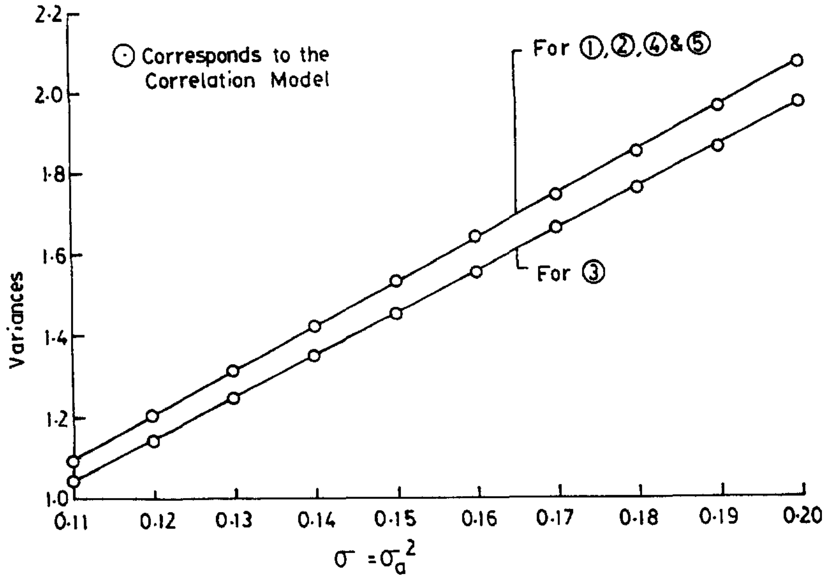


Fig. 1. Var ($\mu_1$) of a Leipholz column for $g^* = 30$ and $\bar{\mu}_1 = 25$ when $E$ is random.

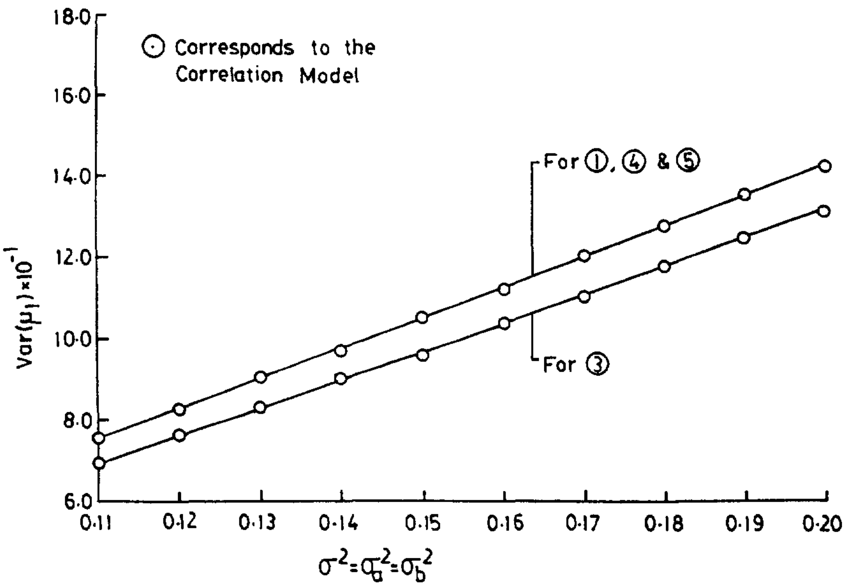


Fig. 2. Var ($\mu_1$) of a Leipholz column for $g^* = 30$ and $\bar{\mu}_1 = 25$ when $E$ and $m$ are random.

that model the uncertain fluctuations of material properties and loadings is brought out through this numerical study. The order of response variability can be seen to be larger when the mass distribution is random.

## 5. CONCLUSIONS

Eigensolution variability of one-dimensional non-self-adjoint structural systems that have an uncertain distribution of material properties and loadings is evaluated based on a stochastic modelling of uncertain parameters. An asymptotic solution to the individual realizations of eigenvalues and eigenvectors of the non-self-adjoint differential equation that describes the free-undamped oscillations of the structural system is obtained. Second order moments of eigenvalues are calculated based on this asymptotic expansion and in terms of the second order probabilistic characteristics of uncertain material properties and loadings. Principles of local averaging are employed to evaluate the second order moments of eigenvalues in practical circumstances wherein full length correlation information about the random fields that model uncertain material properties is not available. Sensitivity of eigenvalues of the non-self-adjoint system to the variations in material properties and loadings is systematically brought out by embedding their corresponding independent perturbation parameters in both the asymptotic expansions for eigenvalues and eigenvectors as well as the partial differential equation. The numerical study encompasses the most commonly observed correlation models and brings out the sensitivity of eigensolutions to the randomness in each of the material properties of the structural system.

That (i) the response variability of the non-self-adjoint structural system is quite sensitive to the randomness in mass density than to the randomness in Young's modulus and (ii) the solid-fluid systems such as fluid-flowing pipes are many a time more sensitive to the uncertainty in respect of material properties and loadings when compared to completely-solid structural systems, can be observed from the covariance expressions of eigenvalues. These facts can be efficiently used in controlling the parameter variability of a mechanical component during the manufacturing stage itself. It is known that for a particular manufacturing material and for a prescribed variability of its strength properties such as Young's modulus, the uncertainty associated with parameters such as mass per unit length can be controlled during the production stages of a structural component. When the reliability of the non-self-adjoint structural system is prescribed, the allowable level of randomness in such parameters can be calculated from the expressions for the second order moments of eigenvalues that are obtained in the present paper. Moreover, it is observed here that uncertainty-sensitive structural systems, in the present case the fluid-flowing pipes, can be identified by evaluating second order moments of eigenvalues. That a probabilistic analysis is essential when such uncertainty-sensitive systems are deployed in important structures such as nuclear structures, gas lines, conduits in hydro-electric power plants etc., is also observed from the present work. Further, following points are observed from the numerical study and these are quite useful for designing reliable and safer mechanical systems: (i) the triangular, finite power white noise and Gaussian correlation models result in almost same values of second order moments of eigenvalues for certain values of the length of correlation, (ii) first-order AR correlation structure results in larger values of eigenvalue moments, (iii) larger values of constants of correlation lead to larger values of eigenvalue variability as far as the triangular and first order AR fields are concerned, (iv) the randomness in mass density is many a time more severe than that in Young's modulus, in amplifying the response variability and (v) the triangular and first-order AR correlation structures lead to almost the same order of response variability amplification and further, the second-order AR correlation function leads to comparatively smaller values of response variability amplification.

The foregoing illustrates an efficient method of effectively integrating the concepts of probability theory, stochastic fields and asymptotic analysis, to analyze and design real life mechanical systems. This method of analysis leads to the systematic extraction of the attributes of response variability and their relative severity in amplifying response variability. It is well known that the vibratory response of non-self-adjoint systems is highly sensitive to internal damping of the structural material. Inclusion of damping in the probabilistic analysis of the present paper however would require a totally different set of asymptotic expansions from that of the present analysis since the velocity term should also be expanded into an asymptotic series and hence warrants a new analysis.

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